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COMMENT

Renormalisation group for DLA and fixed-point distribution

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Abstract. A renormalisation group method is presented to analyse the multifractal structure of the growth probability in the diffusion-limited aggregation (DLA). A renormalisation group transformation is derived for the probability distribution of the growth bond's conductance. After repeated scaling, an 'invariant' distribution is reached as a fixed-point distribution. The growth probability assigned to each growth bond is represented by a random multiplicative process of the cell's growth probability with fixed-point distribution. A hierarchy of generalised dimensions $D(q)$ is calculated and the α–f spectrum is found.

Recently, there has been increasing interest in the problem of geometrical structure in diffusion-limited aggregation (DLA) (Family and Landau 1986, Pynn and Skjeltorp 1985, Pietronero and Tosatti 1986, Stanley and Ostrowsky 1986, Stanley 1986). It is well known that they have a strong measure of self-similarity, which is characterised by the fractal dimension D (Mandelbrot 1982). Halsey *et al* (1986) and Amitrano *et al* (1986) find that the DLA has the multifractal structure. They calculate the growth probability distribution on the perimeter sites of the aggregates numerically and find a hierarchy of generalised dimensions $D(q)$ and an α–f spectrum. Coniglio (1986) proposes a mechanism which generates multifractality, based on a multiplicative process for the hierarchical model of the percolating cluster. Nagatani (1987) presents a real space renormalisation group (RG) method and finds a random multiplicative process of the cell growth probability under the RG transformation. The infinite exponents $D(q)$ and the α–f spectrum are first found from the standpoint of the RG. Gould *et al* (1983) presented a position space renormalisation group method to derive the fractal dimension. Kolb (1987) derived the fractal dimension from a Monte Carlo RG method.

In this comment, we present an improved RG method for the multifractal structure in the DLA. We propose the renormalisation transformation to the probability distribution of the growth bond conductance which is relevant to the conductance of the surface layer. After repeated transformation, an 'invariant' distribution is reached as a fixed-point distribution. In the letter by Nagatani (1987) the configurational average of the conductance of the growth bond was renormalised and approached a fixed-point value. The growth bond conductance was assumed to be a deterministic variable. However, the growth bond conductance will generally be a random variable. In this comment we derive the probability distribution density of the growth bond conductance by using the RG method.

Let us consider a renormalisation procedure for DLA. Cover all the space of the square lattice by cells of edge b, each containing b^2 bonds; an example for $b=2$ is shown in figure 1. After a renormalisation transformation these cells play the role of 'renormalised' bonds. We classify the renormalised bonds into three types: (a) broken bonds which construct the aggregate, (b) growth bonds which are on the perimeter of

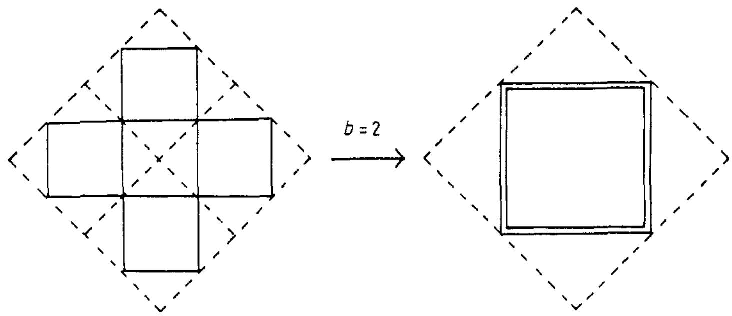

Figure 1. Illustration of the dividing and rescaling of a $b=2$ cell on the square lattice. The rescaled bonds are indicated by double lines. Broken lines indicate boundaries dividing the square lattice into cells.

the aggregate and can be successively grown and (*c*) unbroken bonds which surround the aggregate, except for the growth bonds. If the cell is spanned with the broken bonds then the renormalised bond is considered to be broken (figure 2(*a*)). If the cell is not spanned with the broken bonds and is a nearest neighbour to the cell with a spanning cluster, then the cell is renormalised as the growth bond (figure 2(*b*)). When the cell is constructed by unbroken bonds only and is not a nearest neighbour to the cells with spanning clusters, the cell is renormalised as the unbroken bond (figure 2(*c*)). We concern the growth bond to construct the surface layer of the aggregate. We define the conductance of the growth bond as a conductance of the surface layer. We note that the non-local nature of the electric field is taken into account as the conductance of the growth bond. For later convenience, we summarily explain the renormalisation of the growth bond conductance in Nagatani (1987). We consider a renormalisation of the growth bond conductance by assuming it to be a deterministic variable. If a cell is renormalised as a growth bond, the cell's conductance σ_{n+1} is then represented by the conductance σ_n of the growth bond within the cell after the $(n+1)$th renormalisation transformation: $\sigma_{n+1}=R(\sigma_n)$. The relationship presents the renormalisation group equation. This has a non-trivial solution σ^* (>1). At the fixed point the derivative $\mathrm{d}R/\mathrm{d}\sigma$ has a positive value less than one. This has a stable fixed point. After many repeated renormalisations, the conductance of the growth bond approaches the value σ^* at the fixed point. In this comment we treat the growth bond

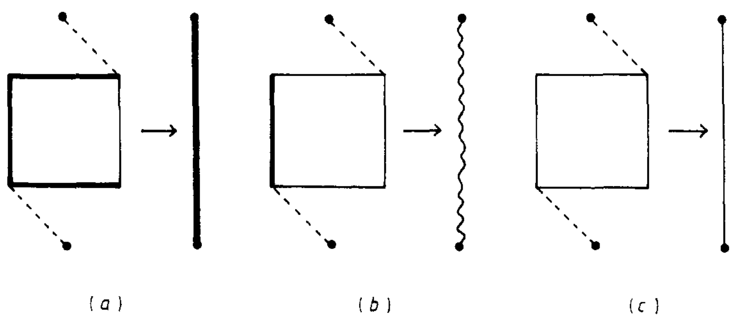

Figure 2. Renormalisation procedure of a $b=2$ cell for DLA. The broken and unbroken bonds are indicated by the bold and light lines, respectively. The bonds, renormalised as the growth bond, are represented by the wavy line. Examples of the distinct configurations are shown in (*a*), (*b*) and (*c*), which are renormalised as the broken, growth and unbroken bonds, respectively.

conductance as a random variable more exactly. We will derive the RG equation for the probability density of the growth bond conductance. An 'invariant' distribution will be reached as a fixed-point distribution.

We consider the growth probability on the growth bond. We define two growth probabilities $\mathbb{P}_i$ and p_i where $\mathbb{P}_i$ indicates the growth probability on the perimeter of the aggregate with size L (where L is the total size of the aggregate) and p_i is the growth probability on the growth bond within the cell. We call p_i the cell's growth probability. Consider the electrostatic problem for cells which can be renormalised as the growth bond. The cell's growth probability p_i on the growth bond i is given by $p_i \sim \sigma_i E_i$ where E_i is the local electric field on the growth bond and σ_i is the conductance of the growth bond. The electric fields on the growth bonds within the cell are determined by the conductance of the growth bonds and the configuration of the cell. All the distinct configurations which it is possible to renormalise as the growth bond are shown in figure 3 for the simplest example ($b=2$). We derive the cell's growth probability. Since the cell's growth probability is independent of the applied voltage of the cell, we apply an arbitrary voltage between the top and the bottom which are represented by the full circle. The broken lines indicate the electrical connections in figure 3. The set of growth probabilities within the cells is given by

$$\begin{aligned}
p_{1,1} &= (\sigma_{1,1}+\sigma_{1,1}\sigma_{1,2})/(\sigma_{1,1}+\sigma_{1,2}+2\sigma_{1,1}\sigma_{1,2}) \\
p_{1,2} &= (\sigma_{1,2}+\sigma_{1,1}\sigma_{1,2})/(\sigma_{1,1}+\sigma_{1,2}+2\sigma_{1,1}\sigma_{1,2}) \\
p_{2,1} &= (\sigma_{2,1}+\sigma_{2,1}\sigma_{2,2})/(\sigma_{2,1}+\sigma_{2,2}+\sigma_{2,1}\sigma_{2,2}) \\
p_{2,2} &= \sigma_{2,2}/(\sigma_{2,1}+\sigma_{2,2}+\sigma_{2,1}\sigma_{2,2}) \\
p_{3,1} &= \sigma_{3,1}/(\sigma_{3,1}+\sigma_{3,2}) \\
p_{3,2} &= \sigma_{3,2}/(\sigma_{3,1}+\sigma_{3,2})
\end{aligned} \tag{1}$$

where p and σ indicate the cell's growth probability and conductance of the growth bond, respectively, and the first and second subscripts label the cell's configuration and position of the growth bonds (see figure 3). We consider the cell's conductance to be possible to renormalise as the growth bond. The conductance of the cell with configurations labelled by (1), (2) and (3) are renormalised as follows:

$$\begin{aligned}
\sigma_1' &= \sigma_{1,1}/(1+\sigma_{1,1})+\sigma_{1,2}/(1+\sigma_{1,2}) \\
\sigma_2' &= \sigma_{2,1}+\sigma_{2,2}/(1+\sigma_{2,2}) \\
\sigma_3' &= \sigma_{3,1}+\sigma_{3,2}.
\end{aligned} \tag{2}$$

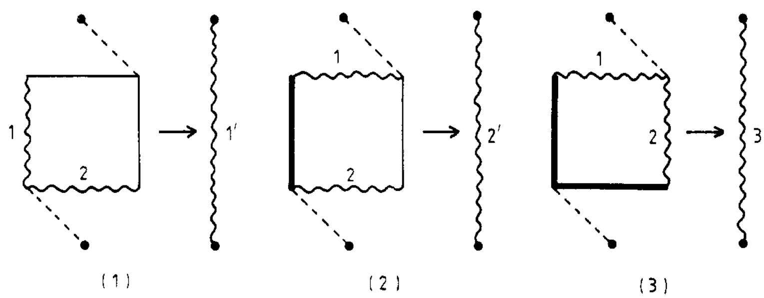

Figure 3. All configurations of the cell being possible to be renormalised as the growth bond. Configuration (2) is obtained by adding a broken bond on the growth bonds 1 or 2 in configuration (1). By adding furthermore a broken bond to configuration (2), configuration (3) occurs.

Consider the configurational probability C_α with which a particular configuration α appears. Figure 3 shows all the configurations of the cell that are possible to renormalise as the growth bond. The distinct configurations are labelled by α ($\alpha = 1, 2, 3$). Configuration (2) is constructed by adding a broken bond to configuration (1). In addition, by adding a broken bond to the growth bond (2, 2) in configuration (2), configuration (3) occurs. The configurational probabilities C_α ($\alpha = 1, 2, 3$) are given by

$$\begin{aligned} C_1 &= 1/(2+p_{2,2}) \\ C_2 &= C_1 \\ C_3 &= p_{2,2}/(2+p_{2,2}). \end{aligned} \tag{3}$$

In general, the renormalised conductance σ'_α of the cell with a particular configuration α is given by

$$\sigma'_\alpha = f_\alpha(\{\sigma_{\alpha,j}\}) \tag{4}$$

where $\{\sigma_{\alpha,j}\}$ represents the set of conductances of the growth bonds within the cell α. The growth probability on the growth bond i within the cell α is represented by a function of the set of growth bond conductances

$$p_{\alpha,i} = g_\alpha(\{\sigma_{\alpha,j}\}). \tag{5}$$

The probability that a given growth cluster configuration occurs is given by the product of growth probabilities of adding a broken bond at each step. The configurational probability C_α is determined by the growth bond conductances of the cells

$$C_\alpha = h_\alpha(\{\{\sigma_{\alpha,j}\}\}) \tag{6}$$

where $\{\{\sigma_{\alpha,j}\}\}$ indicates the set of growth bond conductances for all configurations.

Let us consider the renormalisation transformation of the probability distribution $\rho(\sigma)$ of the growth bond conductance σ. Our procedure is similar to that devised by Stinchcombe and Watson (1976) for percolation conductivity. The probability density $\rho(\sigma)$ that the growth bond has conductance σ is given by

$$\rho(\sigma)^{(n+1)} = \left(\int \sum_\alpha C_\alpha(\{\{\sigma_{\beta,i}\}\})\delta(\sigma - f_\alpha(\{\sigma_{\alpha,i}\}))\right) \prod_\gamma \prod_j \rho(\sigma_{\gamma,j})^{(n)}\, d\sigma_{\gamma,j} \tag{7}$$

where $\rho(\sigma)^{(n+1)}$ indicates the probability density obtained by the $(n+1)$th iteration. For the simplest scaling, the above renormalisation transformation is

$$\begin{aligned} \rho(\sigma)^{(n+1)} = {} & \left(\int\int C_1(\sigma_{2,1}, \sigma_{2,2})\rho(\sigma_{2,1})^{(n)}\rho(\sigma_{2,2})^{(n)}\, d\sigma_{2,1}\, d\sigma_{2,2}\right) \\ & \times \left(\int\int \delta(\sigma - \sigma'_1)\rho(\sigma_{1,1})^{(n)}\rho(\sigma_{1,2})^{(n)}\, d\sigma_{1,1}\, d\sigma_{1,2}\right) \\ & + \left(\int\int C_2(\sigma_{2,1}, \sigma_{2,2})\delta(\sigma - \sigma'_2)\rho(\sigma_{2,1})^{(n)}\rho(\sigma_{2,2})^{(n)}\, d\sigma_{2,1}\, d\sigma_{2,2}\right) \\ & + \left(\int\int C_3(\sigma_{2,1}, \sigma_{2,2})\rho(\sigma_{2,1})^{(n)}\rho(\sigma_{2,2})^{(n)}\, d\sigma_{2,1}\, d\sigma_{2,2}\right) \\ & \times \left(\int\int \delta(\sigma - \sigma'_3)\rho(\sigma_{3,1})^{(n)}\rho(\sigma_{3,2})^{(n)}\, d\sigma_{3,1}\, d\sigma_{3,2}\right). \end{aligned} \tag{8}$$

By inserting the initial distribution $\rho(\sigma)^{(0)}=\delta(\sigma-1)$ into the right-hand side of (8), we obtain a new distribution $\rho(\sigma)^{(1)}$, and we continue so that a sequence $\rho(\sigma)^{(n)}$ of distributions are obtained such that

$$\rho(\sigma)^{(n+1)}=R\{\rho(\sigma)^{(n)}\}. \tag{9}$$

Equation (9) serves as a position space renormalisation group transformation. After many repeated iterations, an 'invariant' distribution is reached:

$$\rho(\sigma)^{*}=\lim_{n\to\infty}\rho(\sigma)^{(n)}. \tag{10}$$

The fixed-point distribution $\rho(\sigma)^{*}$ of the transformation (9) satisfies the relation

$$\rho(\sigma)^{*}=R\{\rho(\sigma)^{*}\}. \tag{11}$$

The integral equation (8) is numerically solved by repeated iteration. Figure 4 shows successive iterations of a conductance distribution. The rapid approach of the sequence $\{\rho(\sigma)^{(n)}\}$ to a limit is found. The fixed-point distribution is shown in figure 5. Under RG transformation, the growth probability $\mathbb{P}_i(L)$ on any growth bond i is given by

$$\mathbb{P}_i(L)=p_{\beta,i}\mathbb{P}_\beta(L/b) \tag{12}$$

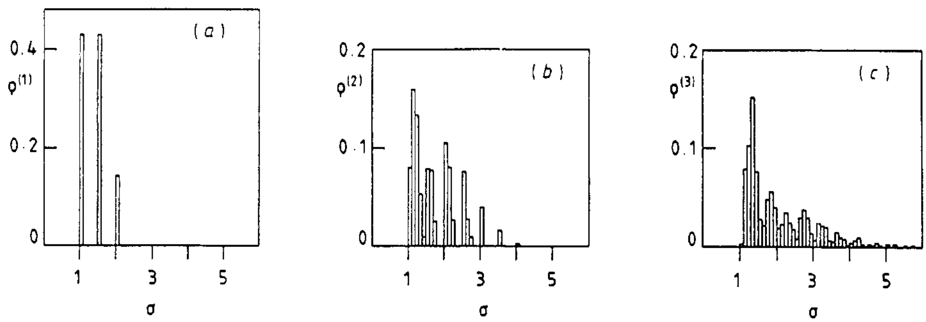

Figure 4. Successive iterations of the conductance distribution under the simple RG transformation. (*a*) The result of the first iteration; (*b*) the result of the second iteration; (*c*) the result of the third iteration.

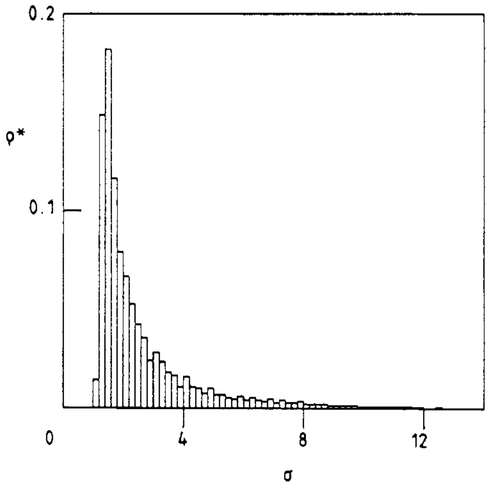

Figure 5. Invariant conductance distribution $\rho(\sigma)^*$ for the RG transformation (8).

where L represents the size of the system, b is the scale factor and $p_{\beta,i}$ indicates the growth probability of the growth bond i within the cell β. After many repeated renormalisations, $p_{\beta,i}$ approaches the random variable $p^*_{\beta,i}$ with the fixed-point distribution of the growth bond's conductance. From relation (12) we can construct an infinite hierarchy of generalised dimensions $D(q) = -(q-1)^{-1} \log(\Sigma_i P_i^q)/\log L$. In the limit of L sufficiently large, the most probable value of ΣP_i^q is given by

$$\left\{\sum_i P_i(L)^q\right\} \sim \left\langle\left\langle\prod_\alpha\left(\sum_i p^q_{\alpha,i}\right)^{C_\alpha}\right\rangle_g\right\rangle_a^n \tag{13}$$

where $\langle\ \rangle_g$ and $\langle\ \rangle_a$ indicate a geometric mean over configurations and an arithmetic mean over the growth bond's conductance of each cell and $n = \log(L)/\log(b)$. For the $b = 2$ scaling, the generalised dimension $D(q)$ is given by

$$D(q) = -(q-1)^{-1}\left[\log\left\langle\left(\sum p^q_{1,i}\right)^{\langle C_1\rangle}\right\rangle + \left\langle C_2 \log\left(\sum p^q_{2,i}\right)\right\rangle + \log\left\langle\left(\sum p^q_{3,i}\right)^{\langle C_3\rangle}\right\rangle\right](\log b)^{-1} \tag{14}$$

where

$$\langle\ \rangle = \int\int \mathrm{d}\sigma_1\, \mathrm{d}\sigma_2\, \rho(\sigma_1)^*\rho(\sigma_2)^*.$$

The exponents $D(q)$ are plotted in figure 6(a). The partition of $D(q)$ into a density of singularities $f(q)$ with singularity strength $\alpha(q)$ is introduced. The relation between α and f is shown in figure 6(b). The α-f spectrum agrees qualitatively with the result of Amitrano *et al* (1986) but is poor quantitatively. This poor result contributes to the small-size cell of the renormalisation transformation. Comparing the present result with that derived from the approximate RG transformation by use of the average conductance (Nagatani 1987), we do not find any difference in the α-f spectrum. The RG transformation of the conductance will be approximated by that of the average conductance proposed by Nagatani (1987).

In summary, we present the renormalisation group method to derive the multifractality of the cluster structure of surface layers in diffusion-limited aggregation. The RG transformation of a probability density of the growth bond conductance is found. After repeated scaling, an 'invariant' distribution is reached as a fixed-point distribution.

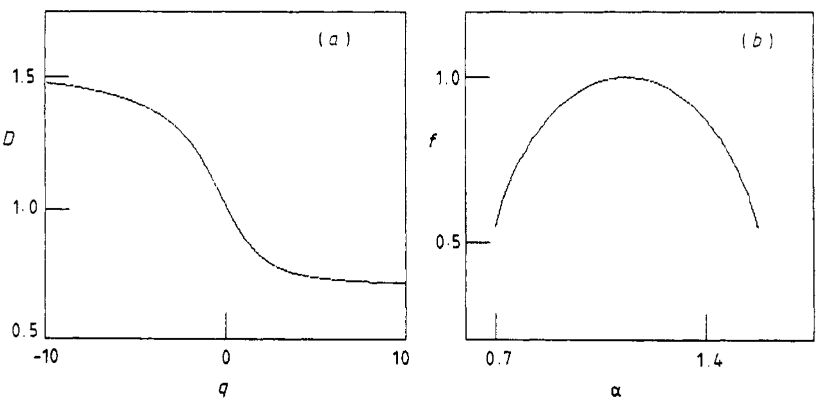

Figure 6. (a) $D(q)$ plotted against q. (b) The plot of f against α.

Under RG transformation the growth probability is represented by a random multiplicative process of the cell's growth probability. It is given by a function of the growth bond conductance which has the fixed-point distribution. The multifractal spectrum is found for DLA. Our RG approach to the scaling structure is general and is not limited to the particular cell considered here.

References

Amitrano C, Coniglio A and di Liberto F 1986 *Phys. Rev. Lett.* **57** 1016
Coniglio A 1986 *Physica* **140A** 51
Family F and Landau D P (ed) 1986 *Kinetics of Aggregation and Gelation* (Amsterdam: Elsevier)
Gould H, Family F and Stanley H E 1983 *Phys. Rev. Lett.* **50** 686
Halsey T, Meakin P and Procaccia I 1986 *Phys. Rev. Lett.* **56** 854
Kolb M 1987 *J. Phys. A: Math. Gen.* **20** L285
Mandelbrot B B 1982 *The Fractal Geometry of Nature* (San Francisco: Freeman)
Nagatani T 1987 *J. Phys. A: Math. Gen.* **20** L381
Pietronero L and Tosatti E (ed) 1986 *Fractals in Physics* (Amsterdam: North-Holland)
Pynn R and Skjeltorp A (ed) 1985 *Scaling Phenomena in Disordered Systems* (New York: Plenum)
Stanley H E (ed) 1986 *Statistical Physics* (Amsterdam: North-Holland)
Stanley H E and Ostrowsky N (ed) 1986 *On Growth and Form* (Dordrecht: Martinus Nijhoff)
Stinchcombe R B and Watson B P 1976 *J. Phys. C: Solid State Phys.* **9** 3221